# Mini-Course on Feynman Integrals

## 1) Introduction

### a) Why Feynman integrals?

Modern applications of QFT ~~require the comput~~ in particle physics and GW physics require the computation of higher-order corrections in perturbative QFT. The main bottle neck here lies in the computation of multi-loop scattering amplitudes and their associated Feynman integrals.

### b) What is this lecture about?

The goal of this lecture series is to provide a compact guide for a modern approach to the computation of Feynman integrals via differential equations. We will discuss, that for any <u>basis</u> of Master integrals $\vec{I}$ we can construct a differential equation of the form

$$\frac{d\vec{I}}{dx_i} = A_i(\vec{x},\epsilon)\,\vec{I}\,, \qquad d = 4-2\epsilon\,, \quad \vec{x} \mathrel{\hat{=}} \text{kinematical scales.} \qquad (1.1)$$

Solving these DE turns out much easier, if we can choose a basis of integrals $\vec{J}$, such that

$$\frac{d\vec{J}}{dx_i} = \epsilon A'_i(\vec{x})\,\vec{J}. \qquad (1.2)$$

The ~~last~~ first lecture will introduce the concept of dimensional Regularization ($d=4-2\epsilon$) and integration-by-parts relations. In lecture two, we will learn how to construct DEs of the form (1.1) and how to solve them. The third lecture shows how to construct a basis transformation $T$, such that $\vec{J} = T\vec{I}$ and we obtain DEs of the form (1.2). We discuss special properties of these integral solutions.

## 2) Dimensional Regularization [hep-ph/0604068]

Why do we need dimensional Regularization?

Consider for example the integral

$$I = \int d^4\ell \, \frac{1}{[\ell^2 - m^2]^2}, \qquad (1.3)$$

for large values of the loop momentum, $\ell^\mu \to \infty$. Then the integral behaves as

$$I \sim \int^\infty d\ell \, \frac{\ell^3}{[\ell^2]^2} = \int^\infty d\ell \, \frac{1}{\ell} = \log(\ell)\Big|^\infty \to \infty. \qquad (1.4)$$

Thus, the integral is divergent and not well defined in $d=4$ dimensions. Therefore, we promote the dimension $d$ to an arbitrary complex parameter, ~~and ta~~ i.e $d = 4-2\epsilon$ and take the limit $d \to 4$ in the end. We still have our usual ations

- **Linearity:**

$$\int d^d\ell \left[\alpha f(\ell) + \beta g(\ell)\right] = \alpha \left[\int d^d\ell \, f(\ell)\right] + \beta \left[\int d^d\ell \, g(\ell)\right] \qquad (1.5)$$

- **Translation Invariance**

$$\int d^d\ell \, f(\ell + p) = \int d^d\ell \, f(\ell) \qquad (1.6)$$

- **Scaling**

$$\int d^d\ell \, f(s\ell) = s^{-d} \int d^d\ell \, f(\ell) \qquad (1.7)$$

- **Lorentz-invariance**

$$\int d^d\ell \, f(\Lambda\ell) = \int d^d\ell \, \det\Lambda^{-1} \, f(\ell) = \int d^d\ell \, f(\ell) \qquad (1.8)$$

Let us now turn to the computation of the integral in Eqn. (1.3) in dim-Reg, but for arbitrary powers of the propagator

$$I_n = \int \frac{d^d\ell}{i\pi^{d/2}} \, \frac{1}{[\ell^2 - m^2]^n}$$

First, we switch to Euclidean momenta such that we can use the $d$-dimensional spherical coordinates. Therefore, we use the following transformation

$$l_E^0 = i l^0\,, \quad \vec{l}_E = \vec{l} \quad \Rightarrow \quad \boxed{l^2 = -l_E^2}\ ?$$

We obtain

$$I_n = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[l^2 - m^2]^n} = (-1)^n i \int \frac{d^d l_E}{i\pi^{d/2}} \frac{1}{[l_E^2 + m^2]^n} = (-1)^n \int d^d\Omega_d \int_0^\infty \frac{d l_E}{\pi^{d/2}} \frac{l_E^{d-1}}{[l_E^2 + m^2]^n}$$

where $\int d^d\Omega_d$ is the area of the $d$-dimensional unit sphere,

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \equiv \Omega_d\,.$$

Then we have

$$I_n = (-1)^n \frac{\Omega_d}{2\pi^{d/2}} \int_0^\infty dl_E^2 \,[l_E^2]^{\frac{d}{2}-1} [l_E^2 + m^2]^{-n} = (-1)^n \frac{\Omega_d}{2\pi^{d/2}} \int_1^0 dx \left(-\frac{m^2}{x^2}\right) \left[m^2 \frac{1-x}{x}\right]^{\frac{d}{2}-1} \left[\frac{m^2}{x}\right]^{-n}$$

$$dl_E = \frac{dl_E^2}{2 l_E} \qquad x = \frac{m^2}{l_E^2 + m^2}$$

$$= (-1)^n \frac{\Omega_d}{2\pi^{d/2}} \int_0^1 dx\, [m^2]^{\frac{d}{2}-n} [1-x]^{\frac{d}{2}-1} [x]^{n-d/2-1}$$

$$\left[ \int_0^1 dt\, t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \right]$$

$$= (-1)^n \frac{\Omega_d}{2\pi^{d/2}} (m^2)^{\frac{d}{2}-n} \frac{\Gamma(\frac{d}{2})\Gamma(n-d/2)}{\Gamma(n)} = (-1)^n (m^2)^{\frac{d}{2}-n} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

Now for $n=2$ and $d = 4 - 2\varepsilon$ we find

$$I_2 = (m^2)^{-\varepsilon} \Gamma(\varepsilon) = \frac{1}{\varepsilon} - \log(m^2) - \gamma_E + O(\varepsilon)\,.$$

where $\gamma_E = 0{,}577216\ldots$ is the Euler-Mascharoni constant. We see that the divergent behaviour is mapped into $1/\varepsilon$ poles in dimensional Regularization.

## 3) Integral Topologies and IBP Relations

In principle one could aim at integrating every single Feynman integral appearing in the scattering amplitudes. In practice this is impossible, as one encounters millions of ~~integrals~~ multi-loop integrals. In the end, we introduce a basis of so-called Master integrals and ~~relate~~ express all occuring integrals as linear combination of these. In order to make these statements more precise we will introduce now the concept of integral families, Master integrals and Integration-by-parts relations.

### 3.1 Integral family

An Integral family is defined for a fixed number of loops and fixed propagators $\rho_i$. However, propagator powers are arbitrary!

$$I[\nu_1, \dots, \nu_N] = \int \frac{d^d l_1}{i\pi^{d/2}} \frac{d^d l_2}{i\pi^{d/2}} \frac{1}{\rho_1^{\nu_1} \dots \rho_N^{\nu_N}} \quad , \quad \rho_i = (l_k - p_i)^2 - m_i^2$$

Beyond one-loop one needs to complete an integral family with auxilliary propagators to obtain a unique map between scalar products and inverse propagators. The number of independent scalar products with loop momenta are given by

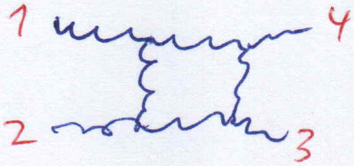
$$N_{SP} = LE + \frac{L(L+1)}{2}$$

L = Number of Loops

E = Number of lin. indep. external momenta

Formally, these Integral families span an infinite-dimensional vectorspace, whose elements are integrals enumerated by $\vec{\nu} = \{\nu_1, \dots, \nu_N\}$. Therefore, it is clear that it is enough to consider a basis of this vectorspace and solve the corresponding Feynman Integrals.

Typically, for a given scattering amplitude one considers the Feynman diagrams with the highest number of propagators to construct Integral families. For four-gluon scattering at one-loop these are Box-Type diagrams

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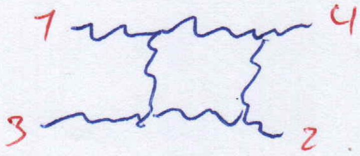



$+ \dots$

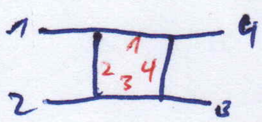
We can define the integral family of the first diagram as

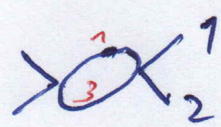
$$I[\nu_1,\nu_2,\nu_3,\nu_4] = \int \frac{d^d\ell}{i\pi^{d/2}} \frac{1}{\rho_1^{\nu_1}\rho_2^{\nu_2}\rho_3^{\nu_3}\rho_4^{\nu_4}}$$

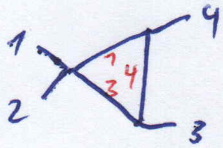
with

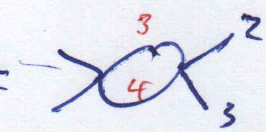
$$\rho_1 = \ell^2, \quad \rho_2 = (\ell-p_1)^2, \quad \rho_3 = (\ell-p_1-p_2)^2, \quad \rho_4 = (\ell-p_1-p_2-p_3)^2.$$

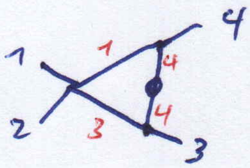
~~This~~ Just a few examples of integrals that are contained in the integral family

$I[1,1,1,1] =$ 


$I[1,0,1,0] =$ 


$I[1,0,1,1] =$ 


$I[0,1,0,1] =$ 


$I[1,0,1,2] =$ 


$I[1,0,0,0] =$ 


Within this Family we have Boxes, Triangles, Bubbles and Tadpole integrals. Higher propagator powers are indicated by additional "dots" on the lines. Tensor integrals have negative propagator powers, e.g. $I[1,-1,1,1]$ but do <u>not</u> have a proper diagramatic Representation. The family is "<u>complete</u>" in the sense we can invert uniquely propagators to obtain scalar products

$$\ell^2 = \rho_1, \quad \ell\cdot p_1 = \frac{1}{2}(\rho_1-\rho_2), \quad \ell\cdot p_2 = \frac{1}{2}(\rho_2-\rho_3+s), \quad \ell\cdot p_3 = \frac{1}{2}(\rho_3-\rho_4-s),$$

where $s = 2(p_1\cdot p_2)$

We now have an efficient way to bookkeep infinite many integrals, however, now the following question arises: How do we identify a basis of <u>Master Integrals</u> for a given integral family? This can be achieved via Integration-by-Parts (IBPs) relations.

## 3.2 Integration-by-part-Relations

In dimensional Regularization the following identity holds

$$\int \frac{d^d l_1}{i\pi^{d/2}} \cdots \frac{d^d l_L}{i\pi^{d/2}} \frac{\partial}{\partial l_i^\mu} \left( \frac{v_j^\mu}{\rho_1^{\nu_1} \cdots \rho_N^{\nu_N}} \right) = 0 ,$$

for any $i \in \{1, \dots, L\}$ and arbitrary vector $v_j^\mu$. This allows to derive Relations between integrals. For instance, let us consider again $I_n = \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[l^2-m^2]^n}$. We can derive

$$\int \frac{d^d l}{i\pi^{d/2}} \frac{\partial}{\partial l^\mu} \left( \frac{l^\mu}{[l^2-m^2]^n} \right) = \int \frac{d^d l}{i\pi^{d/2}} \left( \frac{d}{(l^2-m^2)^n} - 2n \frac{l^2 - m^2 + m^2}{[l^2-m^2]^{n+1}} \right)$$

$$= \int \frac{d^d l}{i\pi^{d/2}} \left[ \frac{d-2n}{(l^2-m^2)^n} - 2nm^2 \frac{1}{(l^2-m^2)^{n+1}} \right] = 0$$

$$\Rightarrow (d-2n) I_n - 2nm^2 I_{n+1} = 0 \Rightarrow \boxed{I_{n+1} = \frac{d-2n}{2nm^2} I_n}$$

so we can systematically relate all integrals $I_n$ to the Master Integral $I_1$! In the absence of the mass the relation reads

$$(d-2n) I_n = 0,$$

which already allows us to drop all massless Tadpoles, since

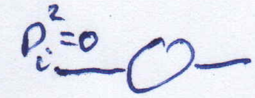
$$I[1,0,0,0] = I[0,1,0,0] = I[0,0,1,0] = I[0,0,0,1] = 0.$$

Similarly we find from

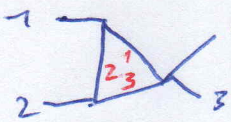
$$\int \frac{d^d \ell}{i\pi^{d/2}} \frac{\partial}{\partial \ell^\mu}\left(\frac{\ell^\mu}{s_1^{\nu_1} s_2^{\nu_2}}\right)=0 \Leftrightarrow (d-2\nu_1-\nu_2)\, I[\nu_1,\nu_2,0,0] - \nu_2\, I[\nu_1-1,\nu_2+1,0,0] = 0$$

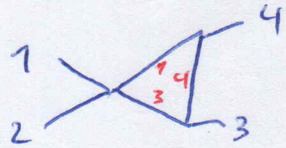
For $\nu_1=\nu_2=1$, we find $I[1,1,0,0]=0$, which we can relate via IBPs to arbitrary $\nu_1, \nu_2$. Therefore,

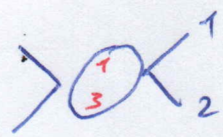
$$I[\nu_1,\nu_2,0,0] = I[0,\nu_2,\nu_3,0] = I[0,0,\nu_3,\nu_4] = I[\nu_1,0,0,\nu_4]$$

$= p_i^2=0$  $=0$

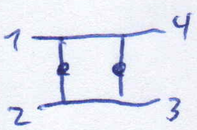
~~Further~~ Furthermore, we find

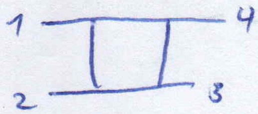


$=$ 

$= -\frac{2(d-3)}{(d-4)s}$ 


In the end, we see that we can express <u>any</u> integral of the integral family in terms of 3 master integrals. For example



$= \frac{(d-5)[s(d-8)-2t]}{st^2}$ 


$- \frac{4(d-8)(d-5)(d-3)}{(d-6)s^2t^2}$ 


$+ \frac{8(d-5)(d-3)}{(d-6)st^3}$ 


These Relations can be systematically computed with public IBP programs. This step of reduction becomes the most complicated step in multi-loop calculations.