4) Differential Equations

Now that we have the concept of integral families and IBP relations, let us construct differential equations. Assume $\vec{I}$ is a vector of master integrals for a given integral family. The integrals depend on external Mandelstam variables s_{ij} and masses m_i^2 that we collectively denote as $\vec{x} = \{s_{ij}, m_i^2\}$. Then the derivatives of the master integrals in x_i are again integrals of the same family. By IBP reduction, we can therefore construct

$$\frac{\partial \vec{I}}{\partial x_i} = A_i(\vec{x}, \varepsilon)\, \vec{I}\,.$$

These differential equations fullfill the following relations

① $$[A_i, A_j] = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$$ "Integrability condition"

② $$\sum x_i A_i = \mathrm{diag}\left([I_1], \dots, [I_N]\right) \frac{1}{2}$$ "Euler Relation"

Mass-dimension

$$[I] = \frac{dL}{2} - N_\nu\,, \quad N_\nu = \sum_{i=1}^{N} \nu_i$$

if we perform a basis transformation $\vec{J} = T\,\vec{I}$ with $T = T(\vec{x}, \varepsilon)$, then we find

$$\frac{\partial \vec{J}}{\partial x_i} = \frac{\partial T}{\partial x_i}\vec{I} + T\frac{\partial \vec{I}}{\partial x_i} = \frac{\partial T}{\partial x_i}\vec{I} + T A_i \vec{I} = \underbrace{\left(\frac{\partial T}{\partial x_i} T^{-1} + T A_i T^{-1}\right)}_{\equiv A_i'} \vec{J}$$

$$\Rightarrow \boxed{A_i'(\vec{x}, \varepsilon) = \frac{\partial T}{\partial x_i} T^{-1} + T A_i T^{-1}}$$

4.1. Differential operators

In order to compute the derivatives of master integrals we have to rewrite derivatives of Mandelstam variables in terms of momenta, according to

$$\frac{\partial}{\partial x_r} = \sum_{i,j} c_{ij}^{(r)}\, p_i^\mu \frac{\partial}{\partial p_j^\mu}\,.$$

Such a decomposition depends only on the external momenta and is independent of the loop order.

Example: Massless Four-point Kinematics

We eliminate p_4 by momentum conservation, such that the linear independent invariants read

$$p_1^2 = p_2^2 = p_3^2 = 0 \ , \ s = 2(p_1 \cdot p_2) \ , \ t = 2(p_2 \cdot p_3) \ , \ -(s+t) = 2(p_1 \cdot p_3)$$

Then we can write an Ansatz

$$\frac{\partial}{\partial t} = \sum_{i,j=1}^{3} c_{ij} \, p_j^\mu \cdot \frac{\partial}{\partial p_i^\mu}$$

and require

$$\frac{\partial}{\partial t}\left[p_1^2\right] = \frac{\partial}{\partial t}\left[p_2^2\right] = \frac{\partial}{\partial t}\left[p_3^2\right] = 0 \ , \ \frac{\partial}{\partial t}\left[2(p_1 \cdot p_2)\right] = 0$$

$$\frac{\partial}{\partial t}\left[2(p_2 \cdot p_3)\right] = 1 \ , \ \frac{\partial}{\partial t}\left[2(p_1 \cdot p_3)\right] = -1$$

The solution is not unique and we find

1) $\frac{\partial}{\partial t}\left[p_1^2\right] = 0 \implies c_{12} s - c_{13}(s+t) = 0$

2) $\frac{\partial}{\partial t}\left[p_2^2\right] = 0 \implies c_{21} s + c_{23} t = 0$

3) $\frac{\partial}{\partial t}\left[p_3^2\right] = 0 \implies c_{32} t - c_{31}(s+t) = 0$

4) $\frac{\partial}{\partial t}\left[2(p_1 \cdot p_2)\right] = 0 \implies c_{11} s + c_{13} t + c_{22} s - c_{23}(s+t) = 0$

5) $\frac{\partial}{\partial t}\left[2(p_2 \cdot p_3)\right] = 1 \implies -c_{21}(s+t) + c_{22} t + c_{31} s + c_{33} t = 1$

6) $\frac{\partial}{\partial t}\left[2(p_1 \cdot p_3)\right] = -1 \implies -c_{11}(s+t) + c_{12} t + c_{32} s - c_{33}(s+t) = -1$

We can find a solution for

$$c_{11} = c_{12} = c_{13} = 0 \quad \text{and} \quad c_{21} = c_{22} = c_{23} = 0$$

which results in

$$\left.\begin{array}{l} 3) \ c_{32} t - c_{31}(s+t) = 0 \\ 4) \ c_{31} s + c_{33} t = 1 \\ 6) \ c_{32} s - c_{33}(s+t) = 1 \end{array}\right\} \implies c_{31} = \frac{1}{2(s+t)} \ , \ c_{32} = \frac{1}{2t} \ , \ c_{33} = \frac{s+2t}{2t(s+t)}$$

$$\implies \frac{\partial}{\partial t} = \left(\frac{1}{2(s+t)} p_1^\mu + \frac{1}{2t} p_2^\mu + \frac{s+2t}{2t(s+t)} p_3^\mu \right) \frac{\partial}{\partial p_3^\mu}$$

$$\implies \frac{\partial}{\partial s} = \frac{1}{s} p_2^\mu \frac{\partial}{\partial p_2^\mu} + \left(-\frac{t(2s+t)}{2s^2(s+t)} p_1^\mu - \frac{2s+t}{2s^2} p_2^\mu - \frac{t}{2s(s+t)} p_3^\mu \right) \frac{\partial}{\partial p_3^\mu}$$

4.2 Construction of Differential Equations

Let us now apply these derivative operators to our one-loop massless Box example, where we choose the masters

$$I_1 = I[1,0,1,0], \quad I_2 = I[0,1,0,1], \quad I_3 = I[1,1,1,1]$$

Using the derivative operator we find

$$\frac{\partial S_1}{\partial t} = \frac{\partial S_2}{\partial t} = \frac{\partial S_3}{\partial t} = 0$$

$$\left(-\frac{\partial S_4}{\partial t}\right) = \frac{S_1}{2(s+t)} + \frac{s\, S_2}{2t(s+t)} + \frac{S_3}{2(s+t)} - \frac{S_4(s+2t)}{2t(s+t)} - \frac{s}{2(s+t)}$$

and therefore

$$\frac{\partial I_3}{\partial t} = \int \frac{d^d\ell}{i\pi^{d/2}} \frac{1}{S_1 S_2 S_3 S_4^2}\left(-\frac{\partial S_4}{\partial t}\right)$$

$$= \frac{1}{2(s+t)} I[0,1,1,2] + \frac{s}{2t(s+t)} I[1,0,1,2] + \frac{1}{2(s+t)} I[1,1,0,2]$$

$$- \frac{s+2t}{2t(s+t)} I[1,1,1,1] - \frac{s}{2(s+t)} I[1,1,1,2]$$

$$\overset{\text{IBP}}{=} -\frac{2(d-3)}{st(s+t)} I_1 + \frac{2(d-3)}{t^2(s+t)} I_2 + \frac{(d-6)s-2t}{2t(s+t)} I_3$$

We can apply this also to I_1 and I_2 and also consider $\frac{\partial}{\partial s}$ to obtain

$$\frac{\partial \vec{I}}{\partial s} = A_s \vec{I} \quad \text{and} \quad \frac{\partial \vec{I}}{\partial t} = A_t \vec{I}$$

with

$$A_s = \begin{pmatrix} \frac{d-4}{2s} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2(d-3)}{s^2(s+t)} & \frac{-2(d-3)}{st(s+t)} & \frac{(d-6)t-2s}{2s(s+t)} \end{pmatrix}$$

$$A_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{d-4}{2t} & 0 \\ \frac{-2(d-3)}{st(s+t)} & \frac{2(d-3)}{t^2(s+t)} & \frac{(d-6)s-2t}{2t(s+t)} \end{pmatrix}$$

4.3 Solving the differential equations

We have obtained a set of differential equations. Here, we want to explicitly solve these. As for any DE, we need boundary constants to fix integration constants. This is typically a highly non-trivial task! Instead, we can use the DE itself to fix the constants. An independent analysis tells us that the integrals should diverge in the limit

$$s \to 0 \quad \text{or} \quad t \to 0$$

but should stay finite for $s+t \to 0$. We continue to simplify the problem by making integrals dimensionless, via

$$J_i = e^{\gamma_E \varepsilon} (-s)^{-\lambda_i} I_i \quad \text{with } \lambda_1 = \lambda_2 = \frac{d-4}{2}, \ \lambda_3 = \frac{d-8}{2}$$

which builds the transformation

$$\vec{J} = T \vec{I}, \quad T = e^{\gamma_E \varepsilon} \cdot \mathrm{diag}\left[(-s)^{-\lambda_1}, (-s)^{-\lambda_2}, (-s)^{-\lambda_3} \right]$$

Then we have

$$B_s = T \cdot A_s \cdot T^{-1} + \frac{\partial T}{\partial s} T^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4-d}{2s} & 0 \\ \frac{2(d-3)}{s+t} & \frac{-2s(d-3)}{t(s+t)} & \frac{1}{s} - \frac{d-4}{2(s+t)} \end{pmatrix}$$

$$B_t = T \cdot A_t \cdot T^{-1} + \frac{\partial T}{\partial t} \cdot T^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{d-4}{2t} & 0 \\ \frac{-2s(d-3)}{t(s+t)} & \frac{2s^2(d-3)}{t^2(s+t)} & \frac{(d-6)s-2t}{2t(s+t)} \end{pmatrix}$$

In addition, we will combine both DEs by introducing

$$t = xs \implies dt = x\,ds + s\,dx$$

$$d\vec{J} = \left(\frac{\partial \vec{J}}{\partial s}\right)_t ds + \left(\frac{\partial \vec{J}}{\partial t}\right)_s dt$$

$$= \left[\left(\frac{\partial \vec{J}}{\partial s}\right)_t + x \left(\frac{\partial \vec{J}}{\partial t}\right)_s \right] ds + \left[s \left(\frac{\partial \vec{J}}{\partial t}\right)_s \right] dx$$

$$\equiv \left(\frac{\partial \vec{J}}{\partial s}\right)_x ds + \left(\frac{\partial \vec{J}}{\partial x}\right)_s dx$$

Explicitly, we find

$$\left(\frac{\partial \vec{J}}{\partial s}\right)_x = 0\,, \quad \left(\frac{\partial \vec{J}}{\partial x}\right)_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{d-4}{2x} & 0 \\ \frac{2(3-d)}{x(1+x)} & \frac{2(d-3)}{x^2(1+x)} & \frac{d-2(3+x)}{2x(1+x)} \end{pmatrix} \vec{J} \equiv A_x \vec{J}$$

First, we observe that the first integral decouples completely. Once we removed the mass dimension the integral becomes a constant. Therefore, the integral has to be solved by direct integration. We find

$$J_1 = e^{\gamma_E \varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}$$
$$= \frac{1}{\varepsilon} + 2 + \left(4 - \frac{\pi^2}{12}\right)\varepsilon + \left(8 - \frac{\pi^2}{6} - \frac{7}{3}\zeta(3)\right)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

For J_2 we can solve the DE by integration

$$\frac{\partial J_2}{\partial x} = \frac{d-4}{2x} J_2 = -\frac{\varepsilon}{x} J_2$$
$$\Rightarrow \frac{dJ_2}{J_2} = -\varepsilon \frac{dx}{x} \Leftrightarrow \log(J_2) = -\varepsilon \log(x) + B \Leftrightarrow \boxed{J_2 = B x^{-\varepsilon}}$$

As J_2 is the same integral as J_1 and only differs by $s \leftrightarrow t$, we can use this to fix the boundary constant.

$$\boxed{J_2\big|_{x=1} = B \overset{!}{=} J_1}$$

Therefore,

$$J_2 = e^{\gamma_E \varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} x^{-\varepsilon}$$
$$= \frac{1}{\varepsilon} + \left[2 - \log(x)\right] + \left[4 - \frac{\pi^2}{12} + \frac{1}{2}\left(-4 + \log(x)\right)\log(x)\right]\varepsilon$$
$$+ \left[8 - \frac{\pi^2}{6} - 4\log(x) + \frac{\pi^2}{12}\log(x) + \log^2(x) - \frac{1}{6}\log^3(x) - \frac{7}{3}\zeta(3)\right]\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

The Bubble integrals were easy enough to integrate with the full ε-dependence. In practice, one solves the DE order-by-order in ε, as we will do now for the box integral.

We expand the integrals ~~via~~ and the DE via

$$\vec{J} = \sum_{n=-2}^{\infty} \varepsilon^n \vec{J}^{(n)} \quad , \quad \frac{\partial \vec{J}}{\partial x} = \left[A_x^{(0)} + \varepsilon A_x^{(1)}\right] \vec{J}$$

By comparing the orders of ε^n we find

$$\boxed{\frac{\partial \vec{J}^{(n)}}{\partial x} = A_x^{(0)} \vec{J}^{(n)} + A_x^{(1)} \vec{J}^{(n-1)}}$$

For the BOX-integral this explicitly reads

$$\underbrace{\frac{\partial J_3^{(n)}}{\partial x} = -\frac{1}{x} J_3^{(n)}}_{\text{Homogenous DE}} + \frac{2 J_2^{(n)} - 2x J_1^{(n)} - x J_3^{(n-1)} - 4 J_2^{(n-1)} + 4x J_1^{(n-1)}}{x^2(1+x)}$$

At any order n, the homogenous solution is $J_3^{(n)} = \frac{1}{x} B^{(n)}$

For $\underline{n=-2}$ with $J_2^{(-2)} = J_1^{(-2)} = J_3^{(-3)} = J_2^{(-3)} = J_1^{(-3)} = 0$, we have

$$\frac{\partial J_3^{(-2)}}{\partial x} = -\frac{1}{x} J_3^{(-2)} \Rightarrow \boxed{J_3^{(-2)} = \frac{1}{x} B^{(-2)}}$$

Currently, we can not fix the constant, but we will be able to do so, once we consider the $\underline{n=-1}$ solution.

For $\underline{n=-1}$ with $J_1^{(-2)} = J_2^{(-2)} = 0$, $J_1^{(-1)} = J_2^{(-1)} = 1$ and $J_3^{(-2)} = \frac{1}{x} B^{(-2)}$, we find

$$\frac{\partial J_3^{(-1)}}{\partial x} = -\frac{1}{x} J_3^{(-1)} + \frac{2 - B^{(-2)}}{x^2} + \frac{B^{(-2)} - 4}{x} + \frac{4 - B^{(-2)}}{1+x}$$

By requiring

$$\left.\frac{\partial J_3}{\partial x}\right|_{x=-1} = \text{finite} \Rightarrow \boxed{B^{(-2)} = 4}$$

The resulting DE can be solved via variation of constants.

We take the Ansatz

$$J_3^{(-1)} = C(x) \frac{1}{x}$$

and find

$$\frac{\partial J_3^{(-1)}}{\partial x} = \cancel{-\frac{1}{x} J_3^{(-1)}} + \frac{1}{x}\frac{\partial C}{\partial x} \overset{!}{=} \cancel{-\frac{1}{x} J_3^{(-1)}} - \frac{2}{x^2} \Rightarrow \frac{\partial C}{\partial x} = -\frac{2}{x} \Rightarrow C = B^{(-1)} - 2\log(x)$$

$$\Rightarrow \boxed{J_3^{(-1)} = \frac{B^{(-1)} - 2\log(x)}{x}}$$

For $\underline{n=0}$

$$\frac{\partial J_3^{(0)}}{\partial x} = -\frac{1}{x} J_3^{(0)} + B^{(-1)}\left[\frac{1}{x} - \frac{1}{x^2} - \frac{1}{1+x}\right]$$

with

$$\left.\frac{\partial J_3}{\partial x}\right|_{x=-1} = \text{fin} \Rightarrow \boxed{B^{(-1)} = 0} \qquad \Rightarrow \boxed{J_3^{(0)} = \frac{B^{(0)}}{x}}$$

Finally, for $\underline{n=1}$ we have

$$\frac{\partial J_3^{(1)}}{\partial x} = -\frac{1}{x} J_3^{(1)} + \frac{1}{1+x}\left[-\frac{B^{(0)}}{x^2} + \frac{\pi^2}{6}\frac{(x-1)}{x^2} + \frac{\log^2(x)}{x^2}\right]$$

Again

$$\left.\frac{\partial J_3}{\partial x}\right|_{x=-1} = \text{fin} \Leftrightarrow \lim_{x\to -1}\left[-\frac{B^{(0)}}{x^2} + \frac{\pi^2}{6}\frac{(x-1)}{x^2} + \frac{\log^2(x)}{x^2}\right] \overset{!}{=} 0 \Rightarrow \boxed{B^{(0)} = -\frac{4\pi^2}{3}}$$

$$\Rightarrow \boxed{J_3^{(1)} = \frac{1}{x}\left[\frac{7\pi^2}{6}\log(x) + \frac{1}{3}\log^3(x) - \log(1+x)\left[\pi^2 + \log^2(x)\right] - 2\log(x)\,\mathrm{Li}_2(-x) + 2\,\mathrm{Li}_3(-x) + B^{(1)}\right]} \qquad \boxed{B^{(1)} = -\frac{34}{3}\zeta(3)}$$

We see that we can generate higher order terms systematically by solving the DE order-by-order in ε. However, the functions and constants ($\mathrm{Li}_n(-x), \zeta(n)$) become more and more complicated.