# 5) Pure Integrals

We have seen that one can construct DE for a basis of Feynman Integrals and solve these order-by-order in $\varepsilon$. At last, we want to discuss how to choose a "good" basis, such that the DE simplifies.

## 5.1 Uniform Transcendality

Let us introduce the concept of "uniform transcendality" (UT). We call an Integral with expansion

$$I_i = \varepsilon^{-m} \sum_{n=0}^{\infty} \varepsilon^n I_i^{(n)}$$

with $m$ a fixed integer and $I_i^{(n)}$ a pure function of transcendent weight $n$, a UT integral. The transcendental weight is defined by

$$T(\text{rational number}) = 0, \quad T(\pi) = 1, \quad T(\zeta(n)) = n$$

$$T(\text{rational function}) = 0, \quad T(\log(x)) = 1, \quad T(\mathrm{Li}_n(x)) = n$$

and the product is defined by

$$T(f_1 f_2) = T(f_1) + T(f_2).$$

Thus, a pure integral has the properties

$$\boxed{T\left(I_i^{(n)}\right) = n, \quad T\left(\frac{\partial I_i^{(n)}}{\partial x}\right) = n-1}$$

Clearly, the integrals we have considered before are not pure!
For a basis of UT integrals the differential equation takes the form

$$\boxed{\frac{\partial \vec{I}}{\partial x_i} = \varepsilon A_i(\vec{x}) \vec{I}} \qquad \text{"canonical DE"} \qquad \text{arxiv: 1304.1806}$$

The integrability condition simplifies to

$$\boxed{[A_i, A_j] = 0, \quad \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right) = 0}$$

The latter condition can be read as path-independence ($\nabla \times \vec{A} = 0$), which tells us that one can write $\vec{A}$ as a gradient field

$$A_i = \frac{\partial}{\partial x_i} \sum_\alpha M_\alpha \log(w_\alpha)$$

functions of kinematics

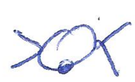
Matrix with rational Numbers

## 5.2 Pure one-loop Basis

At one-loop a general recipe exist to construct a cut basis. We choose the Master integrals according to

| Topology | Dimension | Prefactor |
|---|---|---|
| Tadpole | $2-2\varepsilon$ | $\varepsilon$ |
| Bubble | $2-2\varepsilon$ | $\varepsilon$ |
| Triangle | $4-2\varepsilon$ | $\varepsilon^2$ |
| Box | $4-2\varepsilon$ | $\varepsilon^2$ |
| Pentagon | $6-2\varepsilon$ | $\varepsilon^2$ |

These basis integrals still need to be multiplied by a kinematic prefactor, that we will discuss in the following. However, for practical purposes one would like to use integrals in $d=4-2\varepsilon$. One can show that

$p^2$  $\Big|_{d=2-2\varepsilon} = -\Big($  $+$  $\Big)\Big|_{d=4-2\varepsilon} = \frac{2}{p^2}(1-2\varepsilon)$  $\Big|_{d=4-2\varepsilon}$,

massless Bubble

~~where we have dropped a kinematic dependent factor.~~ ~~rational prefactor~~

For our example we therefore define

$$J_1 = e^{\gamma_E \varepsilon} \underbrace{(-s)^{\varepsilon}}_{\text{Dim.Less}} \underbrace{\frac{2(1-2\varepsilon)}{s}}_{\text{Dim shift}} \underset{\text{Prefactor}}{\varepsilon}\, I_1 \leftarrow d=4-2\varepsilon \text{ master}$$

$$J_2 = e^{\gamma_E \varepsilon} (-s)^{\varepsilon} \frac{2(1-2\varepsilon)}{t}\, \varepsilon\, I_2$$

$$J_3 = e^{\gamma_E} (-s)^{2+\varepsilon}\, \varepsilon^2\, I_3$$

which can be written as $\vec{J} = T\,\vec{I}$

In this basis the DE takes the form

$$B_s = T A_s T^{-1} + \frac{\partial T}{\partial s} T^{-1} = \underbrace{\begin{pmatrix} -\frac{1}{s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/s \end{pmatrix}}_{B_s^{(0)}} + \underbrace{\varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/s & 0 \\ \frac{s}{s+t} & \frac{-s}{s+t} & \frac{1}{s+t} \end{pmatrix}}_{B_s^{(1)}}$$

$$B_t = T A_t T^{-1} + \frac{\partial T}{\partial t} T^{-1} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/t & 0 \\ 0 & 0 & -1/t \end{pmatrix}}_{B_t^{(0)}} + \underbrace{\varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/t & 0 \\ \frac{-s^2}{t(s+t)} & \frac{s^2}{t(s+t)} & \frac{-s}{t(s+t)} \end{pmatrix}}_{B_t^{(1)}}$$

The DE is still not canonical but we can perform a second Transformation $\vec{J}' = T_2 \vec{J}$

$$T_2 = \exp\left(-\int ds\, B_s^{(0)}\right) \exp\left(-\int dt\, B_t^{(0)}\right) = \begin{pmatrix} s & & \\ & t & \\ & & t/s \end{pmatrix},$$

which brings the DE in canonical form.

$$B_s' = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/s & 0 \\ \frac{t}{s(s+t)} & \frac{-1}{s+t} & \frac{1}{s+t} \end{pmatrix}, \quad B_t' = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/t & 0 \\ \frac{-1}{s+t} & \frac{s}{t(s+t)} & \frac{-s}{t(s+t)} \end{pmatrix}$$

Introducing again $t = xs$

$$\left(\frac{\partial \vec{J}'}{\partial s}\right)_x = 0, \quad \left(\frac{\partial \vec{J}'}{\partial x}\right)_s = \varepsilon \left[ \frac{1}{x} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \frac{1}{1+x} \right] \vec{J}'$$

As before $J_1'$ decouples and explicit integration yields

$$J_1' = 2 - \frac{\pi^2}{6}\varepsilon^2 - \frac{14}{3}\zeta(3)\varepsilon^3 - \frac{47}{720}\pi^4\varepsilon^4 + \left[\frac{7}{18}\pi^2\zeta(3) - \frac{62}{5}\zeta(5)\right]\varepsilon^5 + O(\varepsilon^6)$$

$\Rightarrow$ $J_1'$ is a UT integral ?

For $J_2'$ we also find again

$$\frac{\partial J_2'}{\partial x} = -\frac{\varepsilon}{x} J_2' \quad \Rightarrow \quad \boxed{J_2' = B x^{-\varepsilon} = J_1' x^{-\varepsilon}}$$

$$\mathcal{J}_2^1 = 2 - 2\log(x)\,\varepsilon + \left[-\frac{\pi^2}{6} + \log^2(x)\right]\varepsilon^2 + \frac{1}{6}\left[\pi^2\log(x) - 2\log^3(x) - 28\zeta(3)\right]\varepsilon^3 + \dots$$

$\Rightarrow \mathcal{J}_2^1$ is a UT integral!

The biggest difference occurrs for the Box integral. When solving the DE again order-by-order in $\varepsilon$ it reads

$$\frac{\partial \vec{\mathcal{J}}^{(n)}}{\partial x} = A_x^{(1)} \vec{\mathcal{J}}^{(n-1)}$$

which, compared to our first approach, has no homogenous part and can be simply integrated

$$\vec{\mathcal{J}}^{(n)} = \vec{B}^{(n)} + \int dx\, A_x^{(1)} \vec{\mathcal{J}}^{(n-1)}.$$

For the Box-integral we explicitly have

$$\mathcal{J}_3^{(n)} = B^{(n)} + \int dx\, \frac{1}{x}\left[\mathcal{J}_2^{(n-1)} - \mathcal{J}_3^{(n-1)}\right] + \int dx\, \frac{1}{1+x} \underbrace{\left[-\mathcal{J}_1^{(n-1)} - \mathcal{J}_2^{(n-1)} + \mathcal{J}_3^{(n-1)}\right]}_{\text{will always determine } B^{(n-1)}}$$

We find

$n=0$: $\mathcal{J}_3^{(0)} = B^{(0)}$

$n=1$: $$\mathcal{J}_3^{(1)} = B^{(1)} + \int dx\, \frac{1}{x}\left[2 - B^{(0)}\right] + \int dx\, \frac{1}{1+x} \underbrace{\left[-4 + B^{(0)}\right]}_{\Rightarrow B^{(0)} = 4 \text{ finite at } x=-1}$$
$$= B^{(1)} + \int dx\, \frac{-2}{x} = B^{(1)} - 2\log(x)$$

$n=2$: $$\mathcal{J}_3^{(2)} = B^{(2)} + \int dx\, \frac{1}{x}\left[-2\log(x) - B^{(1)} + 2\log(x)\right] + \int dx\, \frac{B^{(1)}}{\underbrace{1+x}_{\Rightarrow B^{(1)} = 0}}$$
$$= B^{(2)}$$

and so on. We find

$$\mathcal{J}_3 = 4 - 2\log(x)\,\varepsilon - \frac{4\pi^2}{3}\varepsilon^2 + \left[\frac{7\pi^2}{6}\log(x) + \frac{1}{3}\log^3(x) - (\pi^2 + \log^2(x))\log(1-x) - 2\log(x)\,\mathrm{Li}_2(-x) + 2\,\mathrm{Li}_3(-x) - \frac{34}{3}\zeta(3)\right]\varepsilon^3 + \dots$$

which is UT!