Introduction to Feynman-Integrals

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Problem 1: Massless Bubble integral

Derive the following solution to the integral in $d = 4 - 2\epsilon$

$$\int \frac{d^d\ell}{i\pi^{d/2}} \frac{1}{[\ell^2]^{\nu_1}[(\ell-p)^2]^{\nu_2}} = (-1)^{\nu_1+\nu_2}(-p^2)^{2-\epsilon-\nu_1-\nu_2} \frac{\Gamma(2-\epsilon-\nu_1)\Gamma(2-\epsilon-\nu_2)\Gamma(\nu_1+\nu_2-2+\epsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(4-2\epsilon-\nu_1-\nu_2)} ,$$

1. Combine the two propagators by using *Feynman Parameters*

   $$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty dx_1 dx_2 \frac{\delta(1-x_1-x_2) x_1^{\alpha-1} x_2^{\beta-1}}{[x_1 A + x_2 B]^{\alpha+\beta}}$$

   and perform the x_2 integration.

2. Complete the square in the denominator, i.e find Q and Δ such that

   $$x_1[\ell^2] + (1-x_1)[(\ell-p)^2] = (\ell-Q)^2 - \Delta .$$

   Here Δ is a generalized mass term, i.e it can only depend on x_1, x_2 and p^2.

3. Shift the loop momentum $\ell \to \ell + Q$

4. Use the result of the lecture to perform the loop momentum integration

   $$\int \frac{d^d\ell}{i\pi^{d/2}} \frac{1}{[\ell^2-\Delta]^n} = (-1)^n (\Delta)^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

5. Integrate the remaining x_1 integral via Euler's beta function

   $$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} .$$

Problem 2: Two-loop massless sunrise integral

Compute the following integral in $d = 4 - 2\epsilon$

$$\int \frac{d^d\ell_1}{i\pi^{d/2}} \frac{d^d\ell_2}{i\pi^{d/2}} \frac{1}{\ell_1^2(\ell_1+\ell_2)^2(\ell_2-p)^2} = -(-p^2)^{1-2\epsilon} \frac{\Gamma^3(1-\epsilon)\Gamma(-1+2\epsilon)}{\Gamma(3-3\epsilon)} .$$

1. Use the result of Problem 1 to perform the ℓ_1 integration.

2. Use the result of Problem 1 again to perform the remaining ℓ_2 integration.

Problem 3: Derivative Operators

Follow the lecture notes and make an Ansatz

$$\frac{\partial}{\partial s} = \sum_{i,j=1} c_{ij} p_i^\mu \frac{\partial}{\partial p_j^\mu}$$

to derive the derivative operator for four-point massless kinematics.
Hint: Your result does not need to match the result of the lecture as it is not unique!

Problem 4: Differential equations

Derive the differential equations

$$\frac{\partial \vec{I}}{\partial s} = A_s(\epsilon, s, t)\vec{I}\ , \qquad \frac{\partial \vec{I}}{\partial t} = A_t(\epsilon, s, t)\vec{I}\ .$$

for the following set of master integrals

$$I_1 = I[1,0,1,0]\ , \qquad I_2 = I[0,1,0,1]\ , \qquad I_3 = I[1,1,1,1]\ .$$

problem 5: canonical form

Construct a basis transformation T, such that for the new basis $\vec{J} = T\vec{I}$ the differential equation is in canonical form, i.e.

$$\frac{\partial \vec{J}}{\partial s} = \epsilon B_s(s,t)\vec{J}\ , \qquad \frac{\partial \vec{J}}{\partial t} = \epsilon B_t(s,t)\vec{J}$$